Conservation equations

Conservation equations are all based on the consideration of the flux of some state variable flowing into and out of some region of the domain. In general the sources and sinks within this region are also considered in arriving at a conservation equation. These fluxes generally depend on position, but they may also depend on the state variable itself, or they may represent fluxes of state variable carried into the region by moving material.

As an example consider a 1-D problem with flux flowing from right to left through some region, and with some distributed sources that are a function of position. We will have a flux in the right-hand side of the region of $\sigma + d\sigma$, balanced by a flux out of the left hand side of $-\sigma$. We will also have a flux due to the distributed source which will be equal to $f(x)dx$, where **dx** is the width of the differential area. If we are considering a time-dependent problem, the balance of fluxes will also include a term that represents ``retention'' of flux. This will manifest as a rate of change of the state variable with time, $\frac{\partial u}{\partial t}dx$. Putting all these pieces together provides the following expression.

$$\frac{\partial \sigma(x)}{\partial x} = f(x) - \frac{\partial u}{\partial t} \qquad (1)$$

This says that neglecting time-dependent effects, the gradient of the flux at some point in the domain is just equal to the source at that point. An imbalance between these two terms will result in a change in the local state variable, manifest in the time-dependent term. If a portion of the distributed source is proportional to the state variable we may include a term in the right-hand side $b(x)u$. If material moving at a velocity $c(x)$ is carrying flux into the region we may also add a term $c(x)\frac{\partial u}{\partial x}$ to the right-hand side.

Equation (gif) is not a differential equation, because it is still expressed in terms of the flux, σ, and the state variable, **u**. To complete the differential equation we need a constitutive relation that relates the flux of state variable to the gradient of the state variable. A simple example of such a constitutive relation is Fourier's Law for heat flow problems. This simply expresses the fact that heat flows down a temperature gradient. The flux of heat flowing from a warm region to a cold region is proportional to the temperature gradient between these two regions, where the constant of proportionality is the usual conductivity of the medium. A generic constitutive relation has the form given in the following expression.

$$\sigma(x) = -k(x)\frac{\partial u}{\partial x} \qquad (2)$$

Combining equations (gif) and (gif) we obtain a second-order differential equation.

$$\frac{\partial}{\partial x}\left(-k(x)\frac{\partial u}{\partial x}\right) + c(x)\frac{\partial u}{\partial x} + b(x)u = f(x) - \frac{\partial u}{\partial t} \qquad (3)$$

For 2- and 3-D domains we will have an analogous equation

$$\nabla \cdot (-k\nabla u) + c\nabla u + bu = f - \frac{\partial u}{\partial t} \qquad (4)$$

where **k**, **u**, **c**, **b** and **f** are functions of position within their respective domains, and the ∇ and $\nabla\cdot$ operators are either 2-dimensional or 3-dimensional gradient and divergence operators.

Thus far we have been dealing with a generic conservation equation. We can apply equation (gif) or (gif) to the primary conservation laws of glaciology by making the identifications in Table 1. For example, the generic state variable, **u**, corresponds to the height of the ice surface for the mass conservation equation, to the velocity vector, $\bar{U}$, for the momentum conservation equation, and to the temperature, **T**, for the energy equation.

Each conservation equation is transformed into a differential equation in terms of its own state variable by the use of a particular constitutive equation corresponding to the generic equation (gif).

For mass conservation this will take the form of the column-averaged flow law which can be expressed for pure flow by the following expression.

$$\sigma = UH = -k\nabla h = \frac{2}{n+2}\left[\frac{\rho g|\nabla h|}{A}\right]^n H^{n+2} \qquad (5)$$

From this expression we can identify the constant of proportionality, **k**, as itself being dependent on the surface gradient, ∇h, and the thickness, **H**.

$$k = \frac{2}{n+2}\left[\frac{\rho g}{A}\right]^n |\nabla h|^{n-1} H^{n+2} \qquad (6)$$

This nonlinear problem must then be solved by an iterative process, whereby an initial uniform distribution of **k** is assumed, a solution for **h** and **H** is obtained, a new nonuniform **k** is obtained from this solution, and the process is repeated until it has converged to a solution. This type of iterative linearized solution is a common technique in dealing with such nonlinear constitutive equations.

For the momentum conservation equation we have a generalized flow law relating stress components to stain-rate components, given by the following expression.

$$\sigma_{ij} = \left[\frac{\dot{\epsilon}^{n-1}}{A^n}\right]\dot{\epsilon}_{ij} \qquad (7)$$

Here $\dot{\epsilon}$ is the usual strain invariant which depends on all the components of the strain-rate tensor, **A** is the ice hardness parameter, and the term in the square brackets represents an effective viscosity. Strain rates are related to the gradients of velocity through the following relationship.

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \qquad (8)$$

Again, because of the nonlinear dependence of the flux-like variable on the gradient of the state variable in this constitutive equation, the solution will require a linearization constant, with an iterative procedure to arrive at a self-consistent solution for the velocity vector.

Finally, energy conservation uses Fourier's Law relating the heat flux to the temperature gradient given by the following expression

$$\sigma = -k\nabla T \qquad (9)$$

where **k** is the thermal conductivity. Because this conductivity can itself be a function of temperature, this equation must also be solved by an iterative process.

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